PROBABILITIES

Niveau(x) concerné(s):	Terminale (et/ou Première)
Notions abordées :	Probabilités, vocabulaire, loi binomiale, dilemme du prisonnier
Nombre de séances :	5
Séquence préparée par :	Julien Mougeot
Sources ***Séance 1***	https://ed.ted.com/lessons/check-your-intuition-the-birthday-problem-david-knuffke
Séance 2	*The Curious Incident of the Dog in the Night-time*, Mark Haddon https://www.youtube.com/watch?v=4Lb-6rxZxx0
Séance 3	https://nrich.maths.org/9646
Séance 4	https://www.youtube.com/watch?v=6YDHBFVIvIs
Séance 5	https://en.wikipedia.org/wiki/Prisoner%27s_dilemma High School Economics 3rd edition, Council for economic education, NY

Descriptif des séances :

Séance 1 : The Birthday problem (découverte d'un problème classique)

- Commencer par présenter le problème. Demander aux élèves un pronostic, discuter… On peut envisager un début de preuve.
- Visionner la vidéo TED-Ed (https://www.youtube.com/watch?v=KtT_cgMzHx8).
- Worksheet #1 à compléter (page 2).

Séance 2 : The Monty Hall Problem (problème célèbre, vocabulaire des probabilités conditionnelles)

- Les élèves lisent à la maison le chapitre 101 du livre *The Curious Incident of the Dog in the Night-time*, de Mark Haddon. On peut toutefois envisager l'activité même si les élèves n'ont pas lu le livre.
- Par groupes de 2, les élèves simulent le Monty Hall Problem (prévoir des gobelets en plastique et des bonbons par exemple) et complètent la worksheet #2 (voir page 3). Même si les résultats fluctuent d'un groupe à l'autre, un bilan pour la classe montre clairement la stratégie à adopter (dès qu'un groupe a réalisé les 20 parties, il vient par exemple compléter une feuille de calcul).
- S'il reste du temps, on peut terminer en visionnant la vidéo de Numberphile (https://www.youtube.com/watch?v=4Lb-6rxZxx0).

Séance 3 : Conditional probabilities (exercises)

- Seuls ou à deux, les élèves font les exercices de la worksheet #3 (page 4).
- À ce stade, il peut être utile de distribuer une fiche de vocabulaire général sur les probabilités, afin de faire un point. Un travail peut être mené en parallèle en Anglais euro sur les expressions idiomatiques (to roll the dice, the die is cast, to load the dice, not playing with a full deck…)

Séance 4 : Galton Board and Binomial Distribution

- Visionner la vidéo (https://www.youtube.com/watch?time_continue=1&v=6YDHBFVIvIs&feature=emb_logo)
- Worksheet #4 à compléter (page 5).

Séance 5 : The Prisoner's dilemma

- Worksheet #5 à compléter (page 6). Pour les jeux, les élèves sont par deux. Il peut être intéressant de laisser les élèves faire plusieurs « rounds » afin d'observer l'évolution des comportements.

Worksheet #1 – The Birthday Problem

Imagine a group of people. How big do you think the group would have to be before there's more than a 50% chance that two people in the group have the same birthday (assuming that there are no twins, no leap year, and equal chance of any day of the year being a birthday)?

1. Check your intuition. Try to guess the correct answer: ....

The first step is to ....
It's easier to calculate **the odds** that ....
The odds of a match and the odds of no match must ....

2. With just two people, the probability that they have different birthdays is .... , or about ....

If a third person joins them, the probability that this new person has a different birthday from those two is ……………, about 0.992. With a fourth person, the probability that all four have different birthdays is ……………………….. , which comes out at around 0.983. And so on.

The answers to these multiplications get steadily smaller. When a twenty-third person enters the room, the final fraction that you multiply by is ………… , and the answer you get drops below 0.5 for the first time, being approximately ………… This is the probability that all 23 people have a different birthday. So, the probability that at least two people share a birthday is .... − .... = ... , just greater than 1/2.

3. A group of 5 people has .... possible pairs, because each of the 5 people can be paired to the other 4, and half of those combinations are redundant, so we divide by 2. $\frac{5 \times 4}{2} =$ ......
By the same reasoning, a group of 23 people has ………… = ………… pairs.

The number of pairs grows .... : it's .... to .... of the number of people in the group.

4. How many people would you need in a group to ensure a 99.9% probability of two of them sharing the same birthday? ....

5. The mathematical study of combinations of objects in groups is known as:
○ Probability ○ Statistics ○ Combinatorics ○ Number theorem

6. The birthday problem exposes our often-poor intuition when it comes to probability. How did your original guess compare to the actual answer for the birthday problem? Why do you think that was?

7. The video ends with the statement "sometimes coincidences aren't as coincidental as they seem." Explain.

Worksheet #2 – The Monty Hall Problem

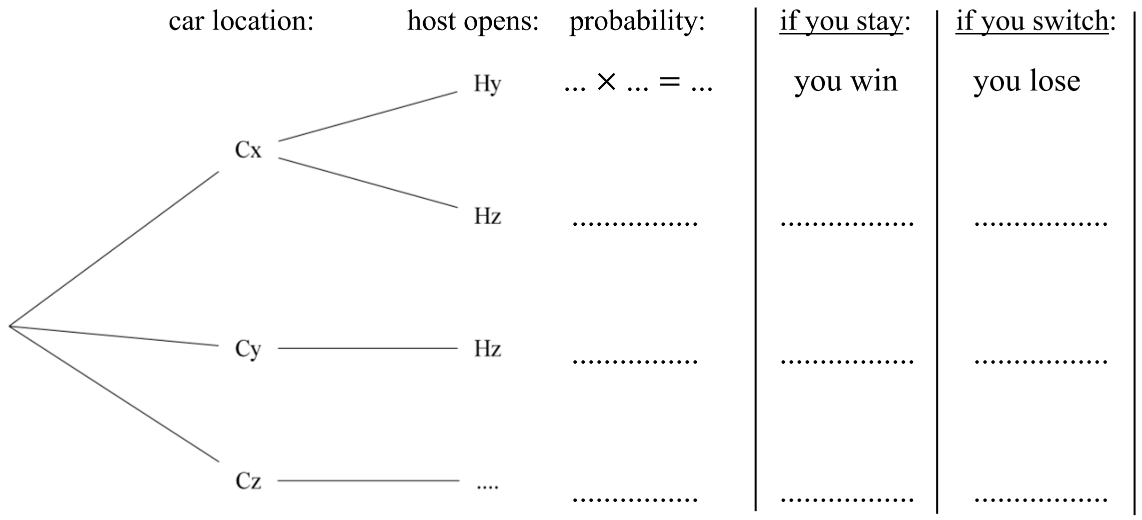
The Monty Hall problem is a famous probability puzzle based on the TV game show Let's Make a Deal and named after its host, Monty Hall. Originally posed in 1975, it became famous when it was presented in Marilyn vos Savant's column in Parade Magazine in 1990. In her column she posed the question:

"Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?"

1. You are now going to simulate the Monty Hall Problem using cups instead of doors. Work in pairs: one person is the host; the other person is the contestant. Each person should be the contestant 10 times and be the host 10 times. Each student should record the results of the games where they are the *contestant*, being sure to switch 5 times and stay 5 times.

Strategy	Wins	Losses	Winning percent-age	Winning percent-age (class)
Switch doors				
Don't switch doors				

2. Let the doors be called x, y and z. Let Cx be the **event** that the car is behind door x and so on.
Let Hx be the event that the host opens door x and so on. Suppose that your initial choice is door x.
Complete the following **tree diagram** (each **branch** is a possible **outcome** and is labelled with a **probability**; the probabilities on groups of branches always add up to 1).



Supposing that you choose door x, the possibility that you win a car if you then switch your choice is given by the formula:

……

P(Hz | Cy) = P(Hz given Cy) is a **conditional probability** (given that the car is behind door y).

$$\mathbf{P(A\ given\ B) = \frac{P(A\ and\ B)}{P(B)}}$$

If your initial choice is door y or door z you will see a similar pattern. In each case, 2/3 of the time one wins by switching, while one wins only 1/3 of the time by staying.

3. Christopher says that "this shows that intuition can sometimes get things wrong. [...] It also shows that [...] numbers are sometimes very complicated and not very straightforward at all". Discuss this statement.

Worksheet #3 – Conditional probabilities

Exercise 1 Why conditional probability is so important

For many students, conditional probability seems to be too hard, and pointless anyway. Frankly, who cares what the chance is of getting a blue ball, given that the previous one was red. But if conditional probability is about the chance that a positive test for cancer means you actually have cancer - that matters. It is a question which can have life and death implications.
A psychologist, Gerd Gigerenzer, in *Reckoning with Risk: Learning to Live with Uncertainty* (Penguin Books, 2002) gives this example:

The probability that a woman of age 40 has breast cancer is about 1 percent. If she has breast cancer, the probability that she tests positive on a screening mammogram is 90 percent. If she does not have breast cancer, the probability that she nevertheless tests positive is 9 percent. What are the chances that a woman who tests positive actually has breast cancer?

1. Without any calculation, try to guess the correct answer: ☐ 90% ☐ 81% ☐ 49.5% ☐ 9%

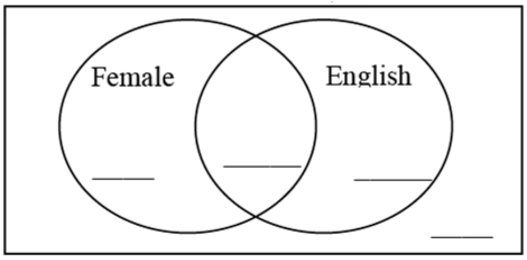
Gigerenzer did research with doctors and lawyers to see how many of them could interpret the data expressed using percentages and probabilities. His results are alarming. That is why it is vital that students need to be helped to understand conditional probability.

2. Using a tree diagram and conditional probability, work out the correct answer.

Exercise 2 Independent events

1. Collect class data to fill in the following two-way table and **Venn diagram**.

	English	Math	Total
Female			
Male			
Total			



2. Suppose that we randomly choose a student from class. Find the following probabilities.
P(Female) = P(not Female) = P(English) =
P(Female AND English) = P(not female AND English) =
P(Female OR English) =

Definition: Two events are **independent** if knowing whether or not one event has occurred does not change the probability that the other event will occur.

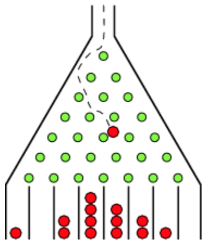
3. What is the probability that a student prefers English, given that they are a female? Write as a percent.

4. What is the probability that a student prefers English, given that they are a male? Write as a percent.

5. Are the events “Female” and “prefers English” independent? Explain.

Worksheet #4 – Galton Board and Binomial Distribution

A "Galton Board" (named after English scientist Sir Francis Galton, 1822-1911) is a triangular array of nails.



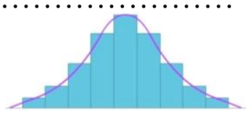
1. Explain how it is used.

Can we accurately predict where any given ball will go? ………

Complete: A ball is …………………………… to end up in the middle than the edge.

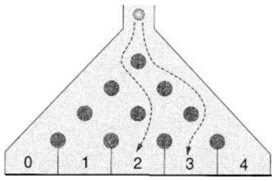
How many paths end up in the leftmost bin? ……

If the number of balls is sufficiently large, we end up with a pattern named ……………………………… also known as the Gaussian distribution (Carl Friedrich Gauss, 1777-1855).



Name two things in nature that follow this pattern: ………………………………

The theoretical curve is a "bell-shaped" curve.



2. In the following problem, there are 4 rows of nails in our Galton Board. We wish to determine the probability distribution of the bins at the bottom (0; 1; 2; 3; 4).

Complete with the words **random variable**, **experiment**, **outcomes**, **event**, **Bernoulli trial**, **binomial distribution**, **success**, **failure**, **parameters**, **independent**:

We consider the ……….. “the ball bounces right”. Each row corresponds to a ……………………., because this ………………… has two possible …………………… : ……….. (the ball bounces right) and …………………… The probability of success is $p =$ ………

X is the …………………………… “Number of right turns from 4 bounces”.

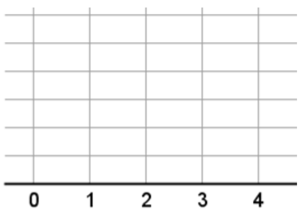
The trials are ……………………., so X has ………………………………………… with …………………… $n = 4$ and $p = 0.5$.

If X ~ B(n,p), the general binomial probability formula is P(X = k) = P(k out of n) = $\binom{n}{k} p^k (1-p)^{n-k}$.

$\binom{n}{k}$ (read “n choose k”) is a **binomial coefficient**. The **expectation** (or **mean value**) is E(X) = np, the **variance** is V(X) = $np(1-p)$ and the **standard deviation** is σ(X) = $\sqrt{np(1-p)}$.

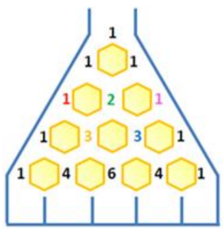
Complete the **probability distribution** and the corresponding bar chart:

k					
P(X = k)					



What is the shape of the chart? …………………………… The expectation of X is E(X) = ……… = …

What do you notice? ……………………………………………………………



Fun fact: Have a look at Pascal's Triangle (Blaise Pascal, 1623-1662). In fact, the Galton Board is just like Pascal's Triangle, with nails instead of numbers. The number on each nail shows you how many different paths can be taken to get to that nail.

Worksheet #5 – The Prisoner's dilemma

Two members of a criminal gang are arrested and imprisoned. Each prisoner is in solitary confinement with no means of communicating with the other. The prosecutors lack sufficient evidence to convict the pair on the principal charge, but they have enough to convict both on a lesser charge. Simultaneously, the prosecutors offer each prisoner a bargain. Each prisoner is given the opportunity either to betray the other by testifying that the other committed the crime, or to cooperate with the other by remaining silent.
The offer is:

- If A and B each betray the other, each of them serves two years in prison
- If A betrays B but B remains silent, A will be set free and B will serve three years in prison (and vice versa)
- If A and B both remain silent, both of them will only serve one year in prison (on the lesser charge).

1. Complete the two-way table which shows the different options and the penalties associated with each.

A \ B	B stays silent	B betrays
A stays silent		
A betrays		-2 / -2

2. What would you do if you were A and you expected B to betray?
Why?

What would you do if you were A and you expected B NOT to betray?
Why?

Explain why this is a dilemma.

3. In-class game: what would YOU do? You have to decide whether to betray your partner, or stay silent.

4. What situations or events in the real world does this game remind you of?

5. Coke and Pepsi are rivals in the soft drink market. Both companies are constantly debating new strategies in order to increase their profits. Each company is independently considering a new advertising campaign. If one company launches a campaign and the other doesn't, this will bring significant rewards. However, advertising is expensive, and if both companies launch new ads, they will cancel each other out and hurt profits. Consider the payoff matrix below. The numbers are daily profits, in millions.

a. What is Pepsi's dominant strategy?
Explain.

		Coke	
		Advertise	**Don't Advertise**
Pepsi	**Advertise**	\$450, \$450	\$1000, \$100
	Don't Advertise	\$100, \$1000	\$600, \$600

b. What is Coke's dominant strategy?

c. What are the two companies most likely to do?

6. A British game show (Golden Balls) created a version of the Prisoner's dilemma. Each of two contestants independently chooses to split or steal the final prize. If both choose split, then the prize is divided evenly. If one chooses split and the other steal, the person who steals gets the entire prize. If both choose steal, however, they both walk away with nothing.

In-class game: let's play "Split or Steal"!